CHAPTER 3 THERMODYNAMIC PRINCIPLES

In this chapter we set out the First and Second Laws of Thermodynamics, which are fundamental in the world of physics, and we examine the dynamics of the main processes encountered as applied to economic systems.

3.1 The First Law of Thermodynamics

The First Law of Thermodynamics is concerned with the principle of conservation of energy applied to systems that undergo changes of state due to transfers of heat and work across a system boundary. The First Law cannot be proved; its validity rests upon the fact that it has never been contradicted by experience. It states: *"When a closed system is taken through a cycle, the net work delivered to the surroundings is proportional to the net heat taken from the surroundings"*. For a *non-flow* gas system where equation (1.1) **PV=NkT** applies, the First Law is generally stated as:

$$Q-W=(U_2-U_1) \qquad (3.1)$$

Where **Q** is the heat passing across the boundary of the system, **W** is the work done or consumed, and **(U_2 – U_1)** is the change in internal energy arising between states 1 and 2. Imagine some gas held in a cylinder by a piston, as in figure 3.1.

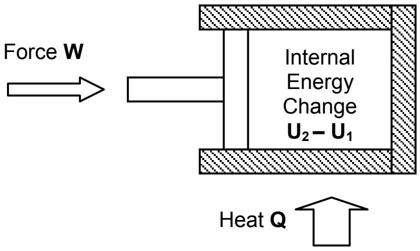

Figure 3.1 Illustration of First Law

If a force **W** is applied to the piston, the gas is compressed, reducing the volume **V** of the cylinder and raising the pressure **P.** The temperature **T** of the gas goes up, with the molecules moving around faster. The gas has therefore accumulated some internal energy (**U_2 – U_1**). Likewise, imagine the piston locked to the cylinder with a pin, holding the volume of the gas constant. Some heat **Q** from outside is applied to the cylinder. The gas gets hot, with the molecules of gas moving around faster, accumulating some internal energy (**U_2 – U_1**). The increased energy of the molecules results in an increased pressure **P** on the cylinder wall and the piston head. Equation (3.1) for the First Law applies.

Similar to equation (3.1), on a unit mass or molecule basis, as in equation (1.2) **Pv=kT**, the First Law for a *non-flow* process is generally expressed as:

$$Q-W=(u_2-u_1) \qquad (3.2)$$

With the lower case letter **u** representing unit mass or molecule internal energy change, being generally referred to as the specific internal energy, similar to the concept of specific volume **v**.

Similar derivations for equations (3.1) and (3.2) can be set out for a thermodynamic *flow* system, though additional components of kinetic energy (the speed of flow of a mass or number of molecules along a pipe or similar), potential energy (energy by virtue of height, for example as in the height of a column of water in a dam) and an extension of internal energy to enthalpy (a wider definition of energy content) are added.

In thermodynamics it is common to consider two process types, reversible and irreversible. In a reversible process, the process is imagined to pass through a continuous series of infinitesimal equilibrium states, such that equation (3.1) can be written in a differential form:

$$dQ - dW = dU \tag{3.3}$$

Where incremental work done **dW** is equal to pressure **P** multiplied by the incremental change in volume **dV**.

$$dW = PdV \tag{3.4}$$

The work done can then be found by summing up all the increments of work. Thus:

$$W = \int_1^2 PdV \tag{3.5}$$

In a reversible process, it is possible for the process to be gradually unwound back through the infinitesimal states to the original position, such that the pressure and volume return to their original values, and the quantities of work done **dW** are reversed. Thus for a reversible process the First Law is stated as:

$$dQ - PdV = dU \tag{3.6}$$

Or in unit mass/molecule terms:

$$dQ - Pdv = du \tag{3.7}$$

where **du** is the incremental change in specific internal energy.

The reversible process therefore is one that cannot be improved upon in thermodynamic terms, as it can be brought back to the starting point without loss. Figure 3.2 illustrates a reversible process.

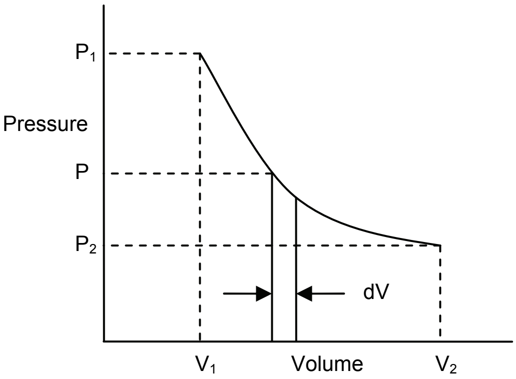

Figure 3.2 Reversible Thermodynamic Process

In an irreversible thermodynamic process, however, a complete return to the starting point would not be possible, and there would always be a difference in one of pressure or volume, if the other was returned to its original position, and there would be a net loss of potential work. In such a case the First Law of Thermodynamics is expressed as in equations (3.1) and (3.2). Figure 3.3 illustrates an irreversible thermodynamic process, in this case a loss of volume.

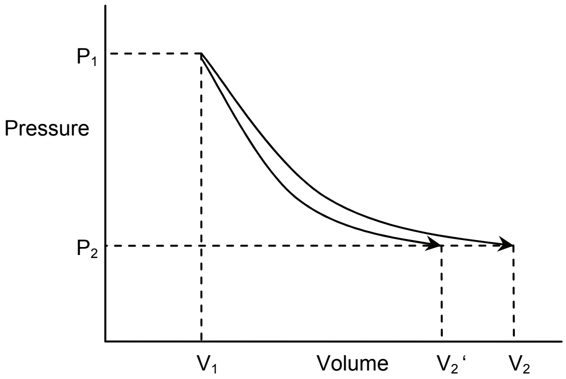

Figure 3.3 Irreversible Thermodynamic Process

For an irreversible *thermodynamic* process, the Work Done **W** is therefore described only by the beginning and end points, and not the intervening process, and only includes processes where there is a change in volume. Thus:

$$W = P_2V_2 - P_1V_1 \tag{3.8}$$

Turning now to our economic system, a similar formulation to the First Law of Thermodynamics can be postulated. We imagine a stock of a good which is fed at one end by work input value of the same good (being a function of price **P** multiplied by volume flow **V** per unit of time), with a similar work output value of the good coming out the other end of the stock, as in figure 3.4.

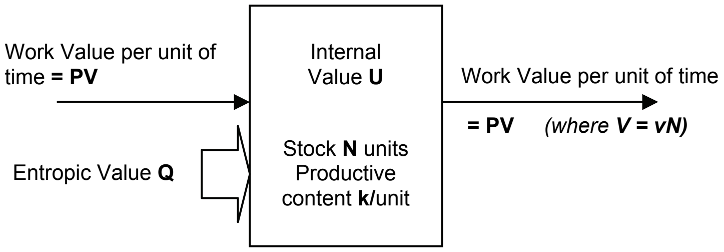

Figure 3.4 An Economic Stock

We imagine four events that can occur to the stock.

First, some *Work Done* **W** per unit of time can occur, being defined as *change* in value per unit of time entering or leaving the system by virtue of a change in volume flow of the *particular* good, *and no other good*. Thus volume flow per unit of time entering *and* leaving the stock could rise from $\mathbf{V_1}$ to $\mathbf{V_2}$. Examples of Work Done would include changes in the volume flow of production output, labour consumption or consumption of raw materials per unit of time.

For a reversible process, incremental Work Done per unit of time would be written as **dW = PdV**, as in equation (3.4); or **Pdv** if analysing on a basis of specific volume rate **v**, as per equation (1.7). For an irreversible process, Work Done **W** would be expressed as $\mathbf{(P_2V_2 - P_1V_1)}$, and in unit stock terms $\mathbf{(P_2v_2 - P_1v_1)}$, the same formats as in equation (3.8).

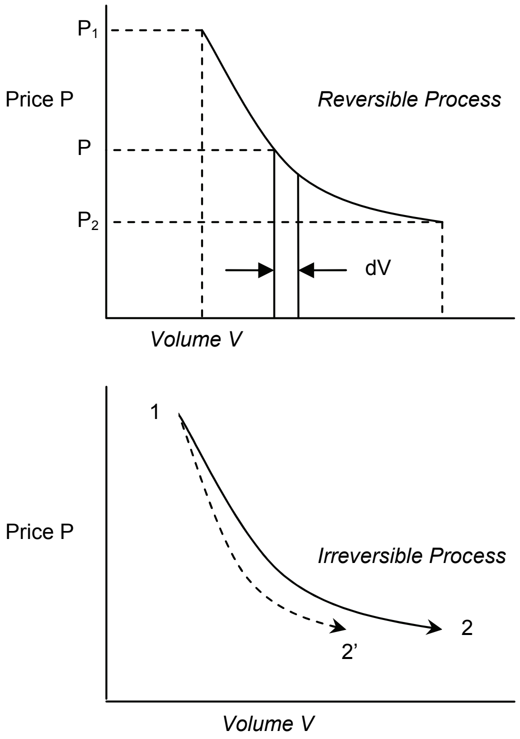

Figure 3.5 Reversible and Irreversible Economic Processes

The concept of reversible and irreversible thermodynamic processes is not readily understood in economics. An economist might argue that by definition one cannot 'undo' a production proceeding from a set of inputs to a set of outputs - at least not easily, and likely at a significant cost. One cannot 'undo' a loaf of bread. It can be argued, however, that some degree of reversibility is achieved through the economic cycle. Thus consumption of producer capital stock in a process results in output which is circulated through the system in return for sales turnover, a part of which consists of profit which can then be used to purchase new capital stock (which has been produced elsewhere) to replace the capital stock used up. This process, as will be seen later, is not a perfect one and losses will always occur. Nevertheless, the reversible process might be viewed as representing an ideal path, albeit not achieved in practice.

The second event that can occur is that value **Q** per unit of time can be put into or taken out of the system, which is *not* represented by a change in volume flow of productive content *of the particular good*. Examples of value **Q** include a scarcity or abundance of the particular good engendered by a change in demand, new money coming into the system, or the consumption of the productive content of *another different good*, some of the value of which can then be *added or transferred* to the particular good via a production process. Value **Q** therefore does *not* represent volume of throughput of the particular good gained or lost. We shall call this the *Entropic Value* added or taken out.

The third event that may occur is a change in the economic internal energy **U** of the stock, which we will call the *'Internal Value'* **U**. There is a difference between the internal value **U** and the stock productive content **Nk**. The latter is fixed by reference to the number of units **N** and the productive content **k** of each unit, which is non-variable. The internal value **U**, on the other hand is a variable value by reference to the value of trading entering and leaving the stock per unit of time.

Imagine a trader with a stock of fashion clothes. The stock has a productive content **Nk** by virtue of the materials from which it is made, and is not a function of price. The trader is in the business of buying and selling such stock. In a good year demand is brisk and he may be able to charge higher prices for his stock. The internal value **U** of the stock is therefore perceived to go up, even though the clothes have not changed in shape or form. If, half way through his trading year, demand suddenly collapses, he may be forced to sell his stock at much lower prices, and not make so many sales. The perceived internal value **U** of the stock therefore goes down.

The internal value **U** of an economic system is therefore a function of the index of trading value **T**, just as internal energy **U** in a thermodynamic system is a function of temperature **T**. The index of trading value **T,** in an economic sense, is a measure of *both* the speed at which economic stocks are being turned over *and* the relative value level (the price) being turned over. There is therefore a connection between the internal energy **U** and the index of trading value **T**, and the internal value **U** of an economic stock therefore has characteristics of flow per transaction time $\mathbf{t_t}$**. T**hus for **N** units in the stock we could write:

$$U = f[C(NT)] \tag{3.9}$$

Where **C** is some function of value or productive content. Likewise, it will be recalled from equations (2.7) and (2.16) in chapter 2 that the ideal economic equation can be written as **P(vN) = NkT**. Thus by substitution into equation (3.9) we have:

$$C = f\left[\frac{Uk}{PvN}\right]$$

Where **P** is price and **v** is the specific volume rate. It will be recalled from equation (2.2) that the specific volume rate **v** is inversely related to the lifetime ratio ξ of a unit of stock and the transaction time $\mathbf{t_t}$ ($\mathbf{v = 1/\xi t_t}$) Moreover, internal value **U** has elements of flow measure per transaction time $\mathbf{t_t}$.. Thus netting out the transaction times we have:

$$C = f\left[\left(\frac{u_t}{P}\right)\xi k\right] \tag{3.10}$$

Where $\mathbf{u_t = U/N}$ is the *Specific Internal Value* per unit of stock for the transaction time $\mathbf{t_t}$**.** If this were taken to be related to the trading price **P** during the transaction time, then the constant **C** would become a function of the productive content and the lifetime ratio:

$$C = f[\xi k] \tag{3.11}$$

We will return to consideration of the function **C** in more depth after a discussion of the Second Law of Thermodynamics. Suffice to say at this point in time that the function **C** in thermodynamic terms is usually called the Specific Heat. In economic terms we will call this the *Specific Value.*

The fourth event that can occur is that the number of stock units **N** may not remain constant. For instance, industrial stocks tend to increase in size as production flow increases. In such cases, there is a difference between the flow of inputs and outputs to the stock, and it might be preferable therefore to work in terms of the shortened ideal economic equation (1.5) **Pv=kT,** where **v=V/N**, or other method, to accommodate the variance.

Thus in economic terms, combining all of the above, according to the First Law of Thermodynamics, a change in the internal value $\mathbf{(U_2 - U_1)}$ will be equal to the addition of work done/consumed **W** and entropic value **Q** entering or leaving the system, depending upon the directions of flow.

The concept of change in internal value is one that can incorporate changes in both volume flow of productive content and entropic value. The following examples are illustrative of the process.

First, we suppose that a seller has a stock of **N** units with productive content **k** each. The seller makes sales of **V** units per year at price **P**. The unit stock turnover is given by the equation **PV=NkT**, as per equation (1.3) chapter 1, where **T** is the index of trading value of the stock. We now suppose that the buyer is prepared to pay an additional amount **Q** per annum for the same unit output. The seller is therefore richer by this additional amount and can for example raise his price **P** to **(P+ΔP)** to match the additional monies. Thus for the same input, the internal value of the seller's stock has risen by **ΔU=ΔQ**, and his index of trading value has risen to **(T+ΔT),** in similar proportion to the rise in price. Thus **ΔT/T = ΔP/P**. It will be noted in this example that no change in volume output has occurred.

As a second example, suppose that instead of the buyer presenting the seller with a present of **ΔQ** per annum, a donor (or a bank) lends the seller an amount **ΔQ** to do with as he pleases, so long as he keeps the money in the business. He can choose to increase his volume flow rate of stock purchases to a rate of **(V + ΔV)**, and hope to sell it onwards at the same price **P**, raising his volume throughput rate by **ΔV**, and increasing his work output by **ΔW** (equals **PΔV**). The internal value of the stock rises by **ΔU**, with a rise in the index of trading value of **ΔT/T = ΔV/V**. It will be noted in this example that no change in output price has occurred.

As a third example, a manufacturer may have facilities to consume additional amounts of *another input (labour, resources etc)* which can be converted via the production process to additional throughput per unit of time of the particular good. There is thus a conversion first of productive content of inputs to value **Q,** and from value **Q** to Work Done **W** on the particular good, accompanied by a possible change in internal value.

The essential points arising from the above development are first, that entropic value **Q,** entering a system of a particular good, does not represent a volume flow of productive content of the particular good; it is essentially money, the promise of money, a difference in asset values, a change in preferences, or value arising from the consumption of *other* goods. Second, the internal value **U** of a stock of a particular good is a measure of the stock value that can be turned around at a certain rate or can be added to, and a rise in internal value can therefore occasion a rise in price, a rise in volume flow or both. Third, there is good reason to call the property **T** an index of trading value, and not just a velocity of circulation, since changes in the index can represent a change in volume flow, a change in price, or both. Last, when connected to a money system, all output goods (which have different levels of productive content from a pin to a house) are then valued with respect to the perceived money productive content $\mathbf{k_M}$, and the index of trading value **T** therefore becomes also a scale of value flow circulation to compare different items of economic value.

It should be strongly emphasised yet again that by stating this we are *not* implying that an absolute scale of monetary 'productive content' can be constructed for a currency, in the manner of weight or energy content. Economics is very much a comparative discipline, and values of currency can and do change. Chapter 5 deals with the application of thermodynamic principles to a monetary stock.

3.2 The Second Law of Thermodynamics and Reversibility

There is nothing implicit in the First Law of Thermodynamics to say that some proportion of heat supplied to an engine must be rejected, and therefore that the cycle efficiency cannot be unity. All that the First Law states is that net work cannot be produced during a cycle without *some* supply of heat, i.e. that a perpetual motion machine of the first kind is impossible.

Likewise in our economic system net work output cannot be achieved without a supply of value arising from input of resources and consumption of some human effort and/or capital stock.

The Second Law of Thermodynamics however is an expression of the fact that some heat must *always* be rejected during a cycle. The law can be stated as: *"It is impossible to construct a system which will operate in a cycle, extract heat from a reservoir, and do an equivalent amount of work on the surroundings.*" There is always some heat left over that cannot be converted into work output.

In economic terms the law could be stated as*: "It is impossible to construct an economic system which will operate in a cycle, extract productive content from a reservoir and do an equivalent amount of work, in terms of productive content, on the surroundings."* There is always a bit of productive content left over that cannot be incorporated into product output. Figure 3.6 sets out a simplified economic cycle to illustrate this point. Input value $\mathbf{Q_1}$ from resources is consumed to produce output **W**, with some waste $\mathbf{Q_2}$ left over. Thus in the cycle, the first law is stated as:

$$W = Q_1 - Q_2 \tag{3.12}$$

And the cycle efficiency is stated as:

$$\eta = \frac{Work\ Done}{Value\ Supplied} = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$\eta = 1 - \frac{Q_2}{Q_1} \qquad (3.13)$$

The work of Ayres and Warr, referred to at chapter 2, showed that the efficiency of exergy processes relating to energy resources, was of the order of 15%, with an overall loss of useful work of 85%. While heat losses in a modern power station are much less than this, it should be remembered that energy resources have to be mined, transported and adapted through a production process, which introduces further losses.

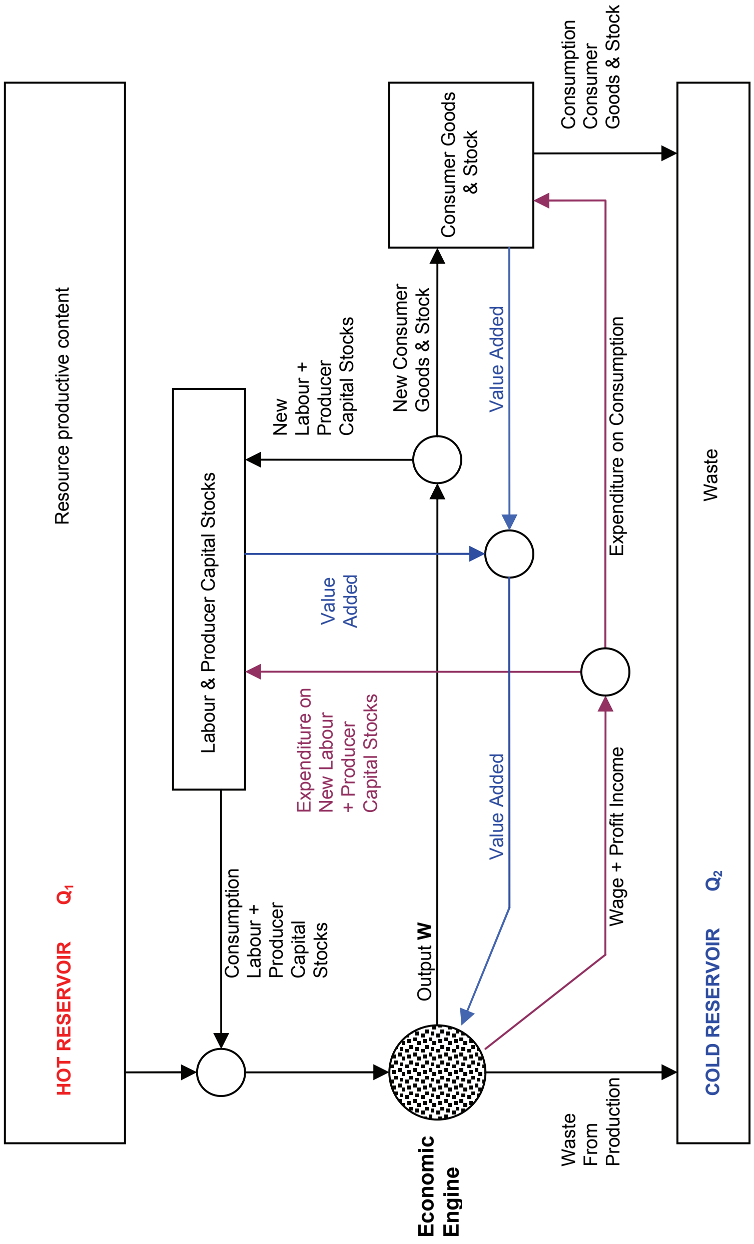

Figure 3.6 A Simple Economic Cycle

The problem however with the structure of an economic system, is that in money terms the system appears to defy the Second Law of Thermodynamics. Thus Value Added of output equals input wages and profits. The system appears to make or conserve money, not consume it.

This apparent non-sequitur arises because much of the productive content included in output has been acquired for no exchange (see figures 2.3 and 2.4). Thus productive content comes free out of the ground or the sea, and from animate and vegetable sources, and then appears as human wage. The economic system does not pay the animal or fish kingdom, and does not pay the earth for its treasures of oil and minerals; it only pays humans and corporations for their part in harvesting and conveying goods to consumers. If the calculation were done on an exergy or other basis of productive content, the efficiency of the human economic system might instead appear wasteful. Thus humans are able to live together off the 'fat of the land', for as long the latter continues to be available and is nurtured by the sun. The moment that potentially such a position no longer becomes an option, through over-population, over-use of resources or pollution, then human economic activity may have to change, perhaps quite dramatically. Samuelson, in his standard textbook on economics, makes the same point about economics including only wages and profits.

Despite the apparent difference between the productive content of resources and the economic value attached to them by humankind, it is reasonable to assume that the economic value added attached at each stage of the process, from mining, fishing and agriculture through to transport, manufacture and services, to final output, is distributed in proportion to the final productive content of the goods included in the economic system, through an input-output model or other means. Thus a thermodynamic interpretation of economic output will have a relation to the original resources. A later chapter deals with construction of economic output cycles to illustrate this.

Reverting back to our thermodynamic analysis; in a closed reversible thermodynamic system there exists a property, such that a change in its value between two states is equal to:

$$S_2 - S_1 = \int_1^2 \left(\frac{dQ}{T} \right)_{rev} \qquad (3.14)$$

Or in differential form:

$$dS = \left(\frac{dQ}{T} \right)_{rev} \quad \text{or} \quad dQ = TdS_{rev} \qquad (3.15)$$

For the unit stock format **Pv=kT**, this would be written as **Tds**, using lower case.

The property **S** is called the *Entropy* of the system, and the value **dS** is the incremental change in entropy. The suffix 'rev' is added as a reminder that when expressed in incremental differential terms the relation holds only for a reversible process.

In thermodynamics, entropy is a property that measures the amount of energy in a physical system that cannot be used to do work. In statistical mechanics it is defined as a measure of the probability that a system would be in such a state, which is usually referred to as the "disorder" or "randomness" present in a system. Given that systems are not in general reversible then, following whatever means are applied to return a system to its starting point, the net change in cycle entropy is commonly stated as:

$$\oint \frac{dQ}{T} \geq 0 \qquad (3.16)$$

Thus to repeat the Second Law; it is impossible to construct a system which will operate in a cycle, extract heat from a reservoir and do an equivalent amount of work on the surroundings. Entropy tends to rise. It is a measure of dispersed value.

Now in our economic system, by combining equation (3.6) for the First law and (3.15) for the Second law and inserting the term for the incremental work done **dW** = **PdV** we have:

$$TdS = dU + PdV \qquad (3.17)$$

And in unit stock terms **(N=1)** we can write:

$$Tds = du + Pdv \qquad (3.18)$$

Equations (3.17) and (3.18) set out the general relations between the properties and, when integrated, give the change in entropy occurring between any two equilibrium states, regardless of whether any particular process joining them is carried out reversibly or not.
Thus to repeat, in our economic system therefore, net entropy change is defined as a measure of the amount of value that is *not* available in a particular economic cycle for conversion into work done.

A series of economic processes is now examined to develop what the concepts mean in economic terms. The key relationships are illustrated at figure 3.7.

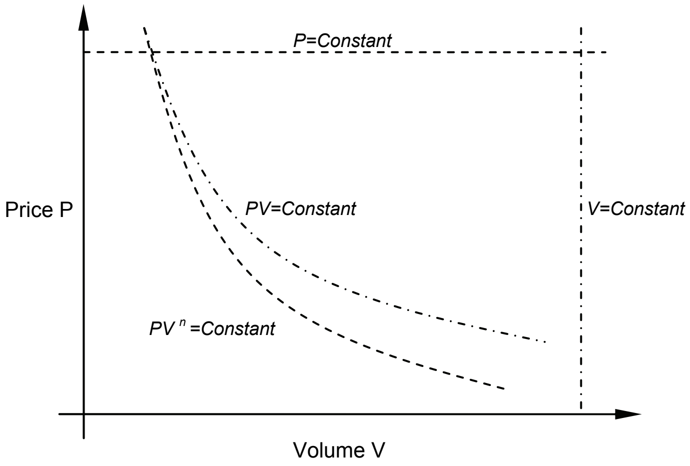

Figure 3.7 Price – volume relationships

3.3 Constant Volume Process

By definition, a constant volume process is one involving no change in volume flow **V** per unit of time, and the work done **PdV** put in or taken out of the economic system is therefore zero. From equation (3.17) we have:

$$TdS = dU + PdV$$

Thence, as **PdV** is zero:

$$TdS = dU \qquad (3.19)$$

Thus a change in the entropy **dS** to the system is reflected only as a change in internal value **dU**. By differentiating the ideal economic equation **PV = NkT** we could write:

$$PdV + VdP = NkdT \qquad (3.20)$$

And remembering in this case that volume change is zero **dV=0**, then:

$$VdP = NkdT$$

Hence by substituting in **PV =NkT** again we have:

$$\frac{dP}{P}=\frac{dT}{T} \qquad (3.21)$$

And:

$$\frac{P_2}{P_1}=\frac{T_2}{T_1} \qquad (3.22)$$

Thus the price of output flow in the process (the flow not having changed) changes exactly in proportion to the change in the index of trading value **dT** arising from the input or output of entropic value **dQ** to and from the system. Nothing has been done to the items in the system, no work has been done; they are just perceived by the players in the system as having more or less value, by virtue of the entropic value **dQ** introduced or taken away. Economists might indicate that a change in price/value of this kind could arise from changes in scarcity or abundance.

Now in order to compute the change in entropy associated with this process, we have first to set out a relationship between the change in the internal value **dU** and the change in the index of trading value **dT**. Similar in structure to equation (3.9), we could write for a single stock unit and for a multiple stock:

$$du = C_v dT \qquad \text{and} \qquad dU = NC_v dT \qquad (3.23)$$

where $\mathbf{C_v}$ is a constant (for an 'ideal' economic system), which we shall call the *Specific Value at constant volume,* being analogous to the specific value **C** developed in equation (3.11). Thus the change in internal value per carrier or unit good goes up in proportion to the change in the index of trading value.

The thermodynamic analogy here is the specific heat at constant volume, being the heat required to raise the temperature of a unit of a gas system by one degree of a scale of temperature. The specific heat of a gas is commonly computed in thermodynamics terms by reference to either unit mass or quantity. The usual measure of the latter is per mole. For a monatomic ideal gas the specific heat at constant volume $\mathbf{C_v = (3/2)\, N_A k}$, where $\mathbf{N_A}$ is Avogadro's Number. Thus specific heat is measured as heat value relating to a multiple of numbers of molecules.

In this book the *Specific Value* of a good at constant volume $\mathbf{C_v}$ in an economic system is defined as the amount of value **du** required to be introduced to the internal value of a unit of stock to change the index of trading value by **dT,** but without any net change in volume flow in or out of the system. It is a measure of ability to store the entropic value that is introduced by **dQ**. In economic terms utility has risen or declined, but nothing of substance has been added or taken away.

It might be supposed that the value $\mathbf{C_v}$ for an economic stock good would be a constant. However, in gas systems, according to the kinetic theory of gases, the specific heat at constant volume is actually dependent upon the complexity of the gas molecules. A simple molecule requires less energy to increase its momentum and raise its temperature, than does a complex one, according to the number of *'degrees of freedom'* – dimensional, rotational and vibrational energies (quantum mechanics introduces yet further degrees of freedom, those of electronic and nuclear). And in reverse, a complex molecule releases more energy for a given drop in temperature than does a simple molecule. The question therefore arises therefore as to whether such a variation is possible in an economic system.

The answer proposed in this book is that variation in $\mathbf{C_v}$ is likely to depend on the 'complexity' of a good in terns of its attributes. In economic terms *'degree of value'* might be a better description than *'degree of freedom'*. To illustrate the concept, money in the form of cash clearly has a nominal value as a means of exchange, with a relatively short lifetime. It might be deemed to have a low *degree of value* $\mathbf{C_v}$. By contrast, both goods that are produced and income generating securities are more complex, containing productive content that can only be released over time.

The two obvious connections therefore, as hinted at equation (3.11), are that the specific value $\mathbf{C_v}$ is a function of both the productive content **k** of the particular good, and the effective lifetime, notated as ξ at equations (2.2) and (3.11). However, some goods, such as gold and diamonds have aesthetic value, adding additional complexity to the valuation. To allow for additional complexity, we choose to define the specific value at constant volume $\mathbf{C_v}$ to be proportional to the embodied value/productive content **k,** and another factor **ω**, which will encompass *both* the lifetime and other aspects. Hence in our constant volume economic system we could write:

$$C_v = \omega k \tag{3.24}$$

Where **ω** might be called the *Value Capacity Coefficient*.

In thermodynamics, the specific heat of a gas is determined by means of highly controlled experiments, to ensure that no heat losses and other factors occur to nullify the results. In economics of course, all values being relative, designing an experiment is likely to be that much more difficult to do. Nevertheless, stating the specific value at constant volume $\mathbf{C_v}$ in this way enables us to continue with the analysis, even if a value cannot immediately be attached.

Now by combining equations (3.15), (3.19), (3.23) and (3.24), the entropy change in the economic process is then written as:

$$dS = \left(\frac{dQ}{T}\right)_{rev} = \left(\frac{dU}{T}\right)_{rev} = NC_v\left(\frac{dT}{T}\right)_{rev} = N\omega k\left(\frac{dT}{T}\right)_{rev} \tag{3.25}$$

Thence by integrating we have, including now for an irreversible system (because the beginning and end points are specified):

$$S_2 - S_1 = N\omega k \ln\left(\frac{T_2}{T_1}\right) \tag{3.26}$$

And by substituting in equation (3.22) we have:

$$S_2 - S_1 = N\omega k \ln\left(\frac{P_2}{P_1}\right) \tag{3.27}$$

which equates the change in entropy in terms of the value capacity coefficient **ω,** and the changes in the index of trading value **T** and price **P**.

In differential form equations (3.26) and (3.27) can be written as:

$$dS = N\omega k\left(\frac{dP}{P}\right)_{rev} = N\omega k\left(\frac{dT}{T}\right)_{rev} \tag{3.28}$$

Thus entropy change in the constant volume process is proportional to the percent change in price or index of trading value. Figure 3.8 illustrates the Constant Volume process, in terms of a **P-V** diagram and an **S-T** diagram.

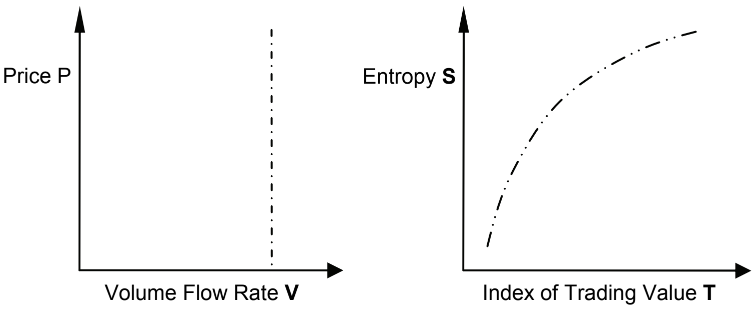

Figure 3.8 Constant Volume Process

By definition, a process that is close to a constant volume process is highly *inelastic*, as a price change can only result in a small change in volume.

3.4 Constant Price Process

This process is suitable for production or consumption stock processes where new units come into the system or go out the other end. By definition, a constant price process is one than can involve a change in volume flow per unit of time, but no change in price occurs. Work done is *not* therefore zero, and any entropic value **dQ** entering or leaving the system must equate to the work done **dW** plus the change in the internal value **dU** of the system. Hence equation (3.17) is stated as:

$$TdS = dU + PdV \qquad (3.29)$$

As with the constant volume process, by differentiating the ideal economic equation **PV=NkT** we have:

$$PdV + VdP = NkdT$$

But since in this process price remains constant and **VdP** is therefore zero, we can write:

$$PdV = NkdT \qquad (3.30)$$

Hence by combining equations (3.23), (3.29) and (3.30) we have:

$$\begin{aligned} TdS &= NC_v dT + PdV \\ &= NC_v dT + NkdT \\ &= NC_p dT \qquad (3.31) \end{aligned}$$

Where $\mathbf{C_p = (C_v + k)}$ is a constant, which we shall call the *Specific Value at constant price*, being analogous to the specific heat at constant pressure in a thermodynamic system, in a similar manner to the constant volume process discussed above.

It will be recalled also from equation (3.24) that the specific value at constant volume using our value capacity coefficient was $\mathbf{C_v = \omega k}$; thence we could write for a constant price process:

$$\begin{aligned} C_p &= \omega k + k \\ &= (\omega + 1)k \qquad (3.32) \end{aligned}$$

The higher value of the specific value $\mathbf{C_p}$ at constant price, compared to that of the specific value $\mathbf{C_v}$ at constant volume, recognises that in adding value to the internal value **U**, volume movement of units takes place. Additional value is flowing through, i.e. not only the entropic value **ωk** (equation 3.28), but also volume of real productive content **k** into and out of the stock.

Now by substituting the ideal equation **PV = NkT** back into equation (3.30) and remembering that price is constant we have:

$$\frac{dV}{V} = \frac{dT}{T} \qquad (3.33)$$

And:

$$\frac{V_2}{V_1} = \frac{T_2}{T_1} \qquad (3.34)$$

Thus the volume flow rate **V** changes exactly in proportion to the change in the index of trading value **T**; which is what one might expect for a constant price process. A change in the index of trading value finds its way wholly into a change in volume, and not price.

Similarly by combining equations (3.15), (3.31) and (3.32), the entropy gain for the process is written as:

$$dS = \left(\frac{dQ}{T}\right)_{rev} = NC_p\left(\frac{dT}{T}\right)_{rev} = Nk(\omega+1)\left(\frac{dT}{T}\right)_{rev} \qquad (3.35)$$

Thence by integrating we have for a constant price process (including for a non-reversible system):

$$S_2 - S_1 = Nk(\omega+1)\ln\left(\frac{T_2}{T_1}\right) \qquad (3.36)$$

And by substituting in equation (3.34) we have:

$$S_2 - S_1 = Nk(\omega+1)\ln\left(\frac{V_2}{V_1}\right) \qquad (3.37)$$

Thus stating the change in entropy for the constant price process in terms of the change in the index of trading value, and in terms of the associated change in volume flow rate. Figure 3.9 illustrates the Constant Price process, in terms of a **P-V** diagram and an **S-T** diagram.

By definition, a process that is close to a constant price process is highly *elastic*, as a small change in price can result in a large change in volume demand.

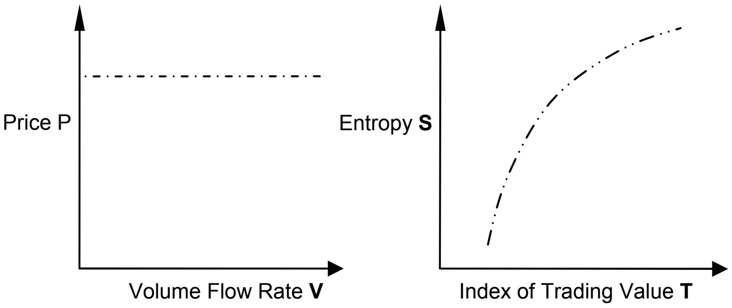

Figure 3.9 Constant Price Process

3.5 Iso-trading Process

As its name suggests, the iso-trading process is one where no change in the index of trading value occurs, that is **dT = 0**. The equivalent thermodynamic process is the isothermal case where temperature change is zero. In mathematical terms we can write:

$$PV = Z \qquad (3.38)$$

where **Z** is a constant, and price varies *inversely* with volume flow. The greater the flow, the lower the price has to be in order for no change in the index of trading value to take place. This formula has common usage in standard textbooks on demand curves, and the shape of the curve is depicted at figure (3.7).

In our model, since the index of trading value **T** is constant, there is no change in incremental internal value **dU** in the stock, and therefore any change in incremental work done **dW** is reflected as a change in entropic value **dQ**. Thus we have:

$$dQ = dW = PdV$$

$$dQ = PdV \qquad (3.39)$$

And differentiating the ideal economic equation **PV=NkT**, we have:

$$PdV + VP = NkdT$$

And since **dT=0**, we have:

$$PdV + VdP = 0$$

Hence:

$$\frac{dP}{P} = -\frac{dV}{V} \qquad (3.40)$$

Indicating, as would be expected, that a change in price is equal and opposite to a change in the volume flow rate.

By substituting equation (3.39) into equation (3.15) for the entropy change we have:

$$dS = \left(\frac{dQ}{T}\right)_{rev} = \frac{1}{T}\left(PdV\right)_{rev} \qquad (3.41)$$

And by further substituting in **PV=NkT**, we have:

$$dS = Nk\left(\frac{dV}{V}\right)_{rev} \qquad (3.42)$$

And

$$dS = -Nk\left(\frac{dP}{P}\right)_{rev} \qquad (3.43)$$

Hence by integrating:

$$S_2 - S_1 = Nk\ln\left(\frac{V_2}{V_1}\right) \qquad (3.44)$$

And:

$$S_2 - S_1 = Nk\ln\left(\frac{P_1}{P_2}\right) \qquad (3.45)$$

Thus we have a logarithmic relationship of entropy change **ΔS** with change in volume flow rate **V**, and an equal and opposite logarithmic relation with change in price **P**. Figure 3.10 illustrates the Iso-trading process, in terms of a **P-V** diagram and an **S-T** diagram.

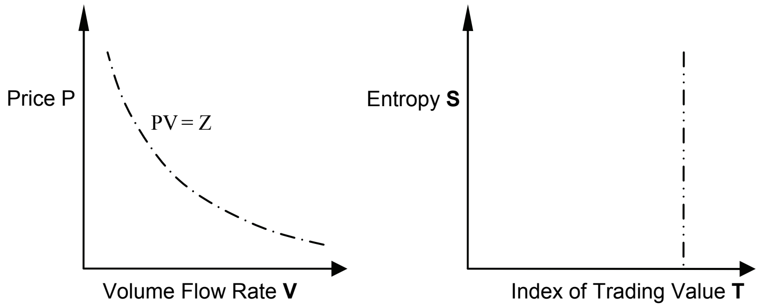

Figure 3.10 Iso-trading Process

By definition, an iso-trading process is one that is neither highly *elastic* nor highly *inelastic*, as a change in price engenders a similar change in volume demand.

3.6 Polytropic Process

A more general type of relationship of price against volume found in economic processes is of the form:

$$PV^n = Z \qquad (3.46)$$

Where **Z** is a constant and **n** is a factor known as the *elastic* index. This is easily confirmed by differentiating the equation to give:

$$\frac{dP}{P} = -n\left(\frac{dV}{V}\right) \qquad (3.47)$$

Supply and demand curves are often drawn to this formula with demand curves having a positive value of **n** and supply curves a negative value. In thermodynamics such processes are called *Polytropic* processes and we shall use the same term here. It will noted that when **n = 0** the relationship reduces to a constant price process, and when **n** = ∞ it reduces to a constant volume one.

Now, referring back to our formula for the work done **W,** we have **PV^n=Z**:

$$W = \int_1^2 PdV = \int_1^2 \frac{Z}{V^n} dV$$

Thence by integration and substitution we obtain:

$$W = \left(\frac{P_2V_2 - P_1V_1}{1-n}\right) \qquad (3.48)$$

And by further substitution of the ideal economic equation **PV=NkT**:

$$W = N\left(\frac{k}{1-n}\right)(T_2 - T_1) \qquad (3.49)$$

Substituting the above back into our equation for the First Law relating entropic value to work done and the change in internal value we obtain:

$$Q - N\left(\frac{k}{1-n}\right)(T_2 - T_1) = U_2 - U_1 \tag{3.50}$$

And re-arranging and substituting in the integrated form of equation (3.23) for the internal value, and equation (3.24) for the specific value at constant volume:

$$Q = NC_v(T_2 - T_1) + N\left(\frac{k}{1-n}\right)(T_2 - T_1)$$

$$= Nk\left(\omega + \frac{1}{1-n}\right)(T_2 - T_1) \tag{3.51}$$

Finally we have an expression for the change in entropy in terms of the index of trading value **T**:

$$S_2 - S_1 = Nk\left(\omega + \frac{1}{1-n}\right)\ln\left(\frac{T_2}{T_1}\right) \tag{3.52}$$

This equation can also be re-stated in terms of changes in price and changes in volume, by substituting in $\mathbf{PV^n = Z}$, although we will not clutter up the picture here. There are nevertheless three expressions relating volume, price and the index of trading value:

$$\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^n \tag{3.53}$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} \tag{3.54}$$

$$\frac{T_2}{T_1} = \left(\frac{V_2}{V_1}\right)^{1-n} \tag{3.55}$$

Figure 3.11 illustrates the polytropic process, in terms of a **P-V** diagram and an **S-T** diagram:

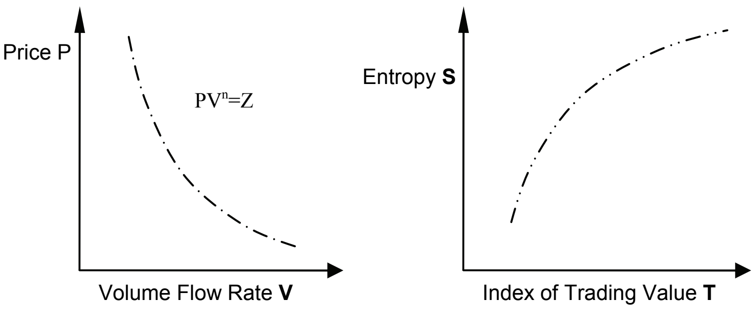

Figure 3.11 Polytropic Process

3.7 Isentropic Process

An important special case of the Polytropic process is the *Isentropic* case, where incremental entropy change **dS** is zero, with no entropic value **Q** entering or leaving the system. Thus in differential form the work done **dW** is equal and opposite to the change in internal value **dU**:

$$dW = -dU = -NC_v dT \tag{3.56}$$

Substituting in **PdV** for **dW** and setting alongside the ideal economic equation **PV=NkT** we have:

$$PdV = -NC_v dT \qquad \text{(First Law)}$$

$$NkdT = PdV + VdP \qquad \text{(Ideal Economic Equation)}$$

Eliminating **dT** from these equations and re-arranging we obtain:

$$0 = \left(1 + \frac{C_v}{k}\right) PdV + \left(\frac{C_v}{k}\right) VdP$$

And since **C_p = (C_v + k)**, this reduces to:

$$0 = C_p PdV + C_v VdP$$

By writing **$C_p/C_v = \gamma = (\omega+1)/\omega$** this then becomes:

$$\gamma \frac{dV}{V} + \frac{dP}{P} = 0$$

And finally by integrating we get:

$$PV^{\gamma} = Z \qquad (3.57)$$

Which is another form of equation (3.46) for the Polytropic process, with the elastic index **$\gamma = (\omega+1)/\omega$** being a function of the value capacity coefficient **ω**.

We can therefore substitute in **γ** for **n** to arrive at the isentropic relationships between volume, price and the index of trading value:

$$\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{\gamma} \qquad (3.58)$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} \qquad (3.59)$$

$$\frac{T_2}{T_1} = \left(\frac{V_2}{V_1}\right)^{1-\gamma} \qquad (3.60)$$

Since by definition there is no change in entropy in this process, all value changes to the internal value of the system involve only changes in real volume and embodied productive content, with no change in entropic value. Figure 3.12 illustrates the Isentropic process, in terms of a **P-V** diagram and an **S-T** diagram.

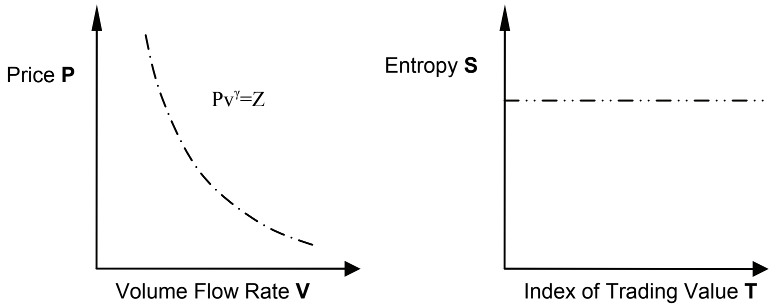

Figure 3.12 Isentropic Process

3.8 Process Entropy

The equations for entropic gain in the preceding processes examined all have a common form, and can all be derived from the entropic gain for the Polytropic process. It will be recalled from equation (3.52) that the expression for the change in entropy was derived as:

$$S_2 - S_1 = Nk\left(\omega + \frac{1}{1-n}\right)\ln\left(\frac{T_2}{T_1}\right) \qquad (3.61)$$

This can be re-stated as:

$$S_2 - S_1 = Nk\lambda \ln\left(\frac{T_2}{T_1}\right) \qquad (3.62)$$

Where

$$\lambda = \left(\omega + \frac{1}{1-n}\right) \qquad (3.63)$$

may be called the *Entropic Index*.

Hence the entropy change for a given process is related to both the change in the index of trading value, and the value of the entropic index **λ**. The latter is a function only of the value capacity coefficient **ω** and of the elastic index **n** of the process. Figure 3.13 shows how the entropic index varies with changes in the elastic index.

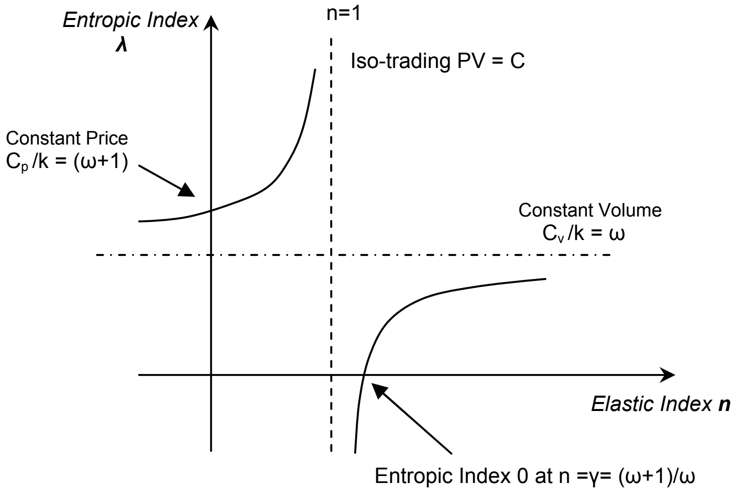

Figure 3.13 Relationship between Entropic Index and Elastic Index

At the point where the elastic index **n** is equal to **(ω+1)/ω),** the entropic index becomes zero with no gain in entropy occurring. This equates to the isentropic case (section 3.7). At the vertical line where the elastic index **n** is equal to 1, we have the iso-trading process (section 3.5) where the entropy gain is not related to change in trading, but only to changes in price and volume. At the horizontal line where the entropic index is equal to **ω** we have the constant volume process (section 3.3). Last, we have that the entropic index **λ** for a constant price process is equal to **(ω+1)** (section 3.4), which implies an elastic index **n** equal to zero, with price not a function of volume.

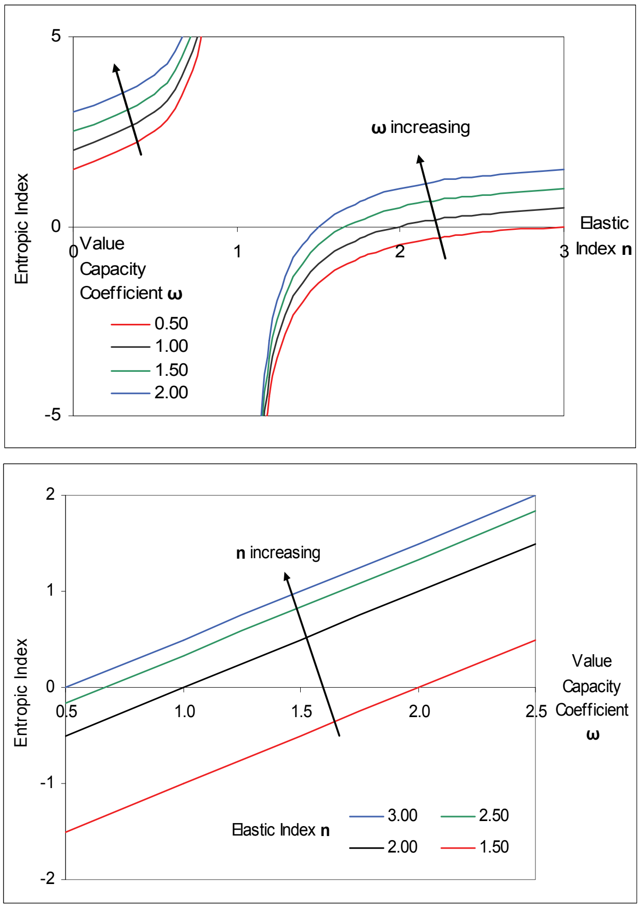

Figure 3.14 Entropic Index, Elastic Index and Value Capacity Coefficient

It can be seen that as the elastic index approaches unity, the curves swing wildly up and down, entailing high positive and negative values of entropic index. Where the line crosses the x-axis the entropic index **λ** becomes zero. This occurs when the elastic index attains the particular value **n=γ**, and is a function of the value capacity coefficient **ω** of the particular good. Figure 3.14 illustrates some values of the entropic index **λ** set against both the elastic index **n** and the value capacity coefficient **ω**.

A further way of looking at the process is to consider deviations of the entropic index **λ** from the isentropic position. It will be recalled from equation (3.63) and figure 3.13 that the condition of nil entropy gain is satisfied when the elastic index **n** is equal to **γ = (ω+1)/ω**. Figure 3.15 sets out a graph of elastic index versus the value capacity coefficient **ω**. The line on the graph represents a locus of points of nil entropy gain where **n = γ = (ω+1)/ω**.

It can be seen that the isentropic elastic index **γ** is low for high values of value capacity coefficient **ω**, and is high for low values of value capacity coefficient **ω**. High values of elastic index are associated with an inelastic position, whereby output volume is not effected much by changes in price; and vice versa.

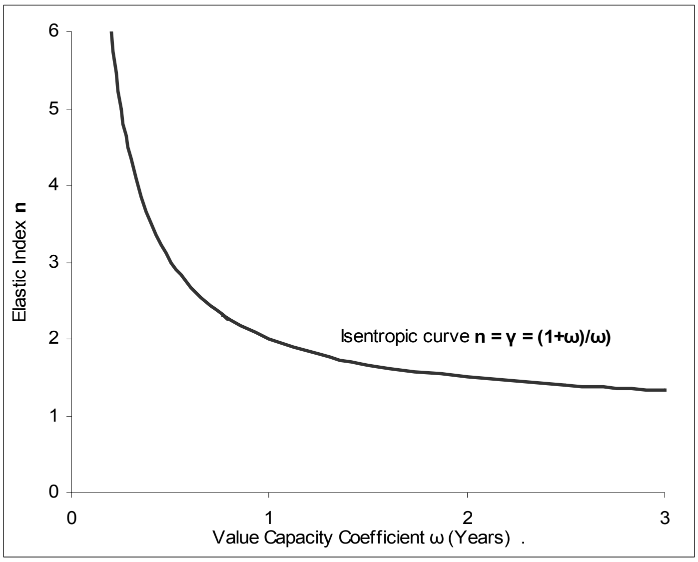

Figure 3.15 Elastic Index as a function of Value Capacity Coefficient

The position for assets and financial instruments such as bonds, with long lifetimes and maturity dates further to the right along the X-axis is therefore different from that for short-term economic assets and instruments such as cash.

A process where the curve shifts either side of the isentropic position, with elastic index **n** greater than or less than the isentropic elastic index **γ**, may reduce or amplify changes in the index of trading value **dT/T**. It can be seen that when the curve shifts there is a change in the effective value of the value capacity coefficient, with potential lifetime being increased or decreased. This point is important when we come to consider a monetary model at chapter 5.

From all of the above, it can be seen that the entropy gain for the general polytropic process is of a logarithmic form, as per the general equation (3.61). This is augmented both by the value capacity coefficient **ω** and the elastic index **n**, as shown in figure 3.16.

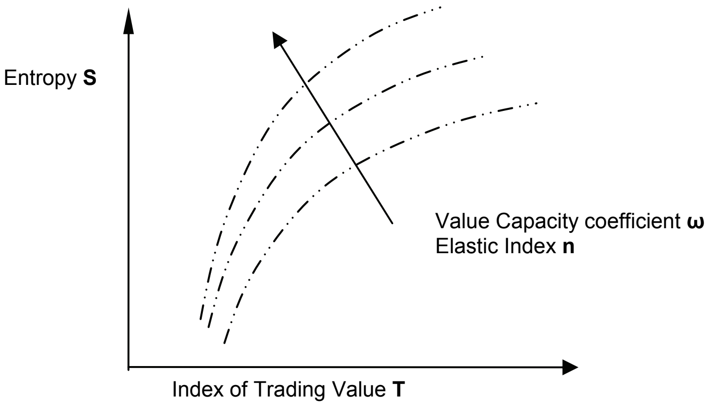

Figure 3.16 Entropy Polytropic Process

The shape of this property should be borne in mind when considering the next section.

3.9 Thermodynamics and Utility

A number of researchers have highlighted similarities between economic utility theory and thermodynamic concepts, in particular entropy. Candeal, Miguel et al (2001) describe a similarity between the utility representation problem in utility theory and the entropy representation problem related to the second Law of Thermodynamics. Sousa and Domingos (2005, 2006) describe a number of aspects of both utility theory and thermodynamics. Smith & Foley (2002, 2004) highlight similarities between utility and entropy. From the theory developed in this chapter some similarities between utility and entropy also occur, perhaps being even more strikingly apparent.

Before proceeding further, it should be noted that in economics it is common to use the character **U** to signify utility. However, since the same character is used in thermodynamics to signify the internal energy of a gas, which we have used also here in connection with the internal value held in a stock by virtue of its throughput; to avoid confusion we will change the economic utility character to **Y**.

The economic concept of utility **Y** is one that is not readily understood and appreciated by scientists, and some basic explanation is necessary to provide a bridge between the two disciplines. In simple terms, utility theory posits that consumers can choose between consuming more or less of a variety of goods, according to their particular circumstances and preferences. Consumers are said subconsciously and individually to attach particular utility values to each good, and endeavour to maximise their total utility within an overall income/budget constraint. Economists use such constructs as indifference curve maps and Edgeworth boxes to show how one consumer's preferences are reconciled to another's. One might express utility **Y** as:

$$Y = f(V_a, V_b \ldots . V_n)$$
$$\text{subject to } (P_a V_a + P_b V_b + \ldots\ldots. P_n V_n) \leq m \qquad (3.64)$$

Where **m** is a budget output value constraint. Utility, in economic parlance, is therefore couched in terms of a value set of mix of volumes of goods, within an overall output value **∑PV≤m.** For a stock process, therefore, utility is related also to the index of trading value **T**, since **PV=NkT.**

A basic tenet of utility theory is that as consumers consume more of the same good, their total utility with respect to that good rises, but grows slower and slower as their level of satisfaction causes them to turn to other possibilities on which to spend their income. Thus marginal utility falls with increasing volume.

An essential point to note is that both the utility and the price attached to any one good may vary, *irrespective of the productive content of each good.* The Law of Diminishing Marginal Utility therefore states that at consumer equilibrium the marginal utility of one good with respect to volume change **∂Y/∂V** divided by its price **P** is equal to the marginal utility of another good with respect to volume change divided by its price:

$$\frac{\left(\partial Y / \partial V\right)_a}{P_a} = \frac{\left(\partial Y / \partial V\right)_b}{P_b} = \ldots . = \frac{\left(\partial Y / \partial V\right)_n}{P_n} \qquad (3.65)$$

This presents the position of maximum utility within the budget constraint at equation (3.64). The law constitutes a basic input to the law of downward sloping demand curves – price being inversely a function of volume, as per most of the curves in figure 3.7 and subsequently developed in this chapter.

A number of utility functions are in common use in economics, including constant elasticity of substitution, isoelastic, Cobb-Douglas, exponential and linear, to explain particular preferences. The isoelastic case for instance is of the form $\mathbf{Y=f(X)=(X)^{1-a}/1-a}$, which, for a value of **a=1,** in the limit reduces to **Y=ln(X)**, the familiar log curve often used in illustrations of utility.

A scientist will readily appreciate from equation (3.65), that no reference is made to the underlying productive content/embodied value, energy or other scientific measure of a good, only the price **P** and the utility value **Y** attached to it by consumers. These are both variables, and their value, according to economic theory, depends upon the views of consumers and supply and demand only.

We have therefore to investigate whether a link between utility theory and thermodynamics can be established, using the theoretic structure set out so far in this chapter, and what differences may exist between the two disciplines.

Before proceeding however, it should be noted that immediately before a choice decision, our consumer has *not* fulfilled his/her utility; and this remains unsatisfied, free and positive, locked into his supply of money (or promise thereof), until the consumer has bought his/her product in exchange for money and consumption of the good(s) has then occurred. A utility value therefore exists with the consumer immediately *before* the point of purchase, and reduces on purchase and once consumption begins to take place. Entropy, on the other hand, is a property that is released from a product as it goes through a process or cycle. For example, if fuel is burnt, useful exergy value is consumed and lost forever, and entropy is released to the ecosystem as a Second Law loss. The difference therefore is that utility declines with the *purchaser or 'owner'*, and entropy declines with the *purchased* or *'product'* (being released to the ecosystem). Utility (and associated budget) therefore might be defined as a *potential entropy value* imagined by humans rather than a formal entropy value.

A further point to consider is that utility decisions occur continuously, as output value flow proceeds. Thus changes in both utility and entropic value in an economic context occur over time. In the case of entropy, this is through the index of trading value **T**.

As a first requirement, any thermodynamic process considered must be able to provide the impetus to allow price and utility to vary quite independently of the underlying productive content. From the First Law of Thermodynamics, the work done **W** alone will not provide such an impetus, since it is concerned *only* with changes in volume associated with real productive content. The most likely candidate therefore is the entropic value **Q** added or taken away, since that accounts for abundance and scarcity, changes in demand and new money, all of which can affect the price of output, and influence consumer preferences.

Reverting to our thermodynamic development, the polytropic case $\mathbf{PV^n = Z}$ represents a suitable process to examine as, from figure 3.13, all of the other processes can be derived from this. The First Law equation for a polytropic economic process, set out at equation (3.51), is equal to:

$$Q = Nk\left(\omega + \frac{1}{1-n}\right)\left(T_2 - T_1\right) \tag{3.66}$$

Where **Q** is the Entropic value being added or taken away during the process. In differential terms this is set out as:

$$dQ = Nk\left(\omega + \frac{1}{1-n}\right)dT \tag{3.67}$$

By dividing both sides by incremental work done **dW (=PdV)** we could re-state this as:

$$\frac{dQ}{PdV} = Nk\left(\omega + \frac{1}{1-n}\right)\left(\frac{dT}{PdV}\right)$$

Substituting in the ideal economic equation **PV=NkT**, we have:

$$\frac{dQ}{PdV} = \left(\omega + \frac{1}{1-n}\right)\left(\frac{dT}{T}\right)\left(\frac{V}{dV}\right) \tag{3.68}$$

And from equation (3.60) for the polytropic process we have in differential form:

$$\frac{dT}{T} = \left(1-n\right)\frac{dV}{V}$$

Substituting this into equation (3.68) and re-arranging we have:

$$\frac{dQ/dV}{P} = \left(\omega + \frac{1}{1-n}\right)(1-n)\left(\frac{dV}{V}\right)\left(\frac{V}{dV}\right)$$

And by further re-arrangement we finish with:

$$\left(\frac{dQ/dV}{P}\right) = (\omega - \omega n + 1) \qquad (3.69)$$

In the alternative we can write:

$$\left(\frac{dQ}{dW}\right) = (\omega - \omega n + 1) \qquad (3.70)$$

Equations (3.69) and (3.70) say:

- Incremental change in Entropic Value **dQ** with respect to incremental volume change **dV**, all divided by price **P** is equal to a simple function of the value capacity coefficient **ω** and the elastic index **n**, and in the alternative:

- Incremental change in Entropic Value **dQ** with respect to incremental change in work done **dW** is equal to the same function of the value capacity coefficient **ω** and the elastic index **n.**

These equations are of considerable importance as, when compared to equation (3.65) for the Law of Diminishing Marginal Utility, they imply that marginal utility **∂Y** might be equivalent to incremental change in entropic value **dQ**, which is also a function of the index of trading value **T** multiplied by the incremental change in entropy **dS**; i.e. **dQ=TdS**.

This is *not* the same, however, as saying that marginal utility is purely a function of entropy change. The index of trading value **T** comes into play as well because utility is based on a stream of value flow **PV**. Moreover, from equation (3.69) it can be seen that values for the right hand side are dependent upon the type of process considered. For the five processes set out in this chapter, the right hand side of equation (3.69) is defined as follows:

Process	***Elastic Index***	***Marginal Entropic Ratio (ω – ωn +1)***
• *Constant volume*	n=∞	-∞ (no volume change)
• *Constant price*	n=0	(ω + 1)
• *Iso-trading*	n=1	1
• *Polytropic*	n=n	(ω – ωn + 1)
• *Isentropic*	n=γ	No value as entropy change is zero

Thus comparisons of entropy value changes for processes with different elastic indices **n** and value capacity coefficients **ω** will yield different answers. We will call the factor (ω – ωn + 1) the *Marginal Entropic Ratio,* being equal to marginal entropic value with respect to work done **dQ/dW**; equal also to **TdS/PdV.** Figure 3.17 sets out a graphical representation of the Marginal Entropic Ratio to explain the differences.

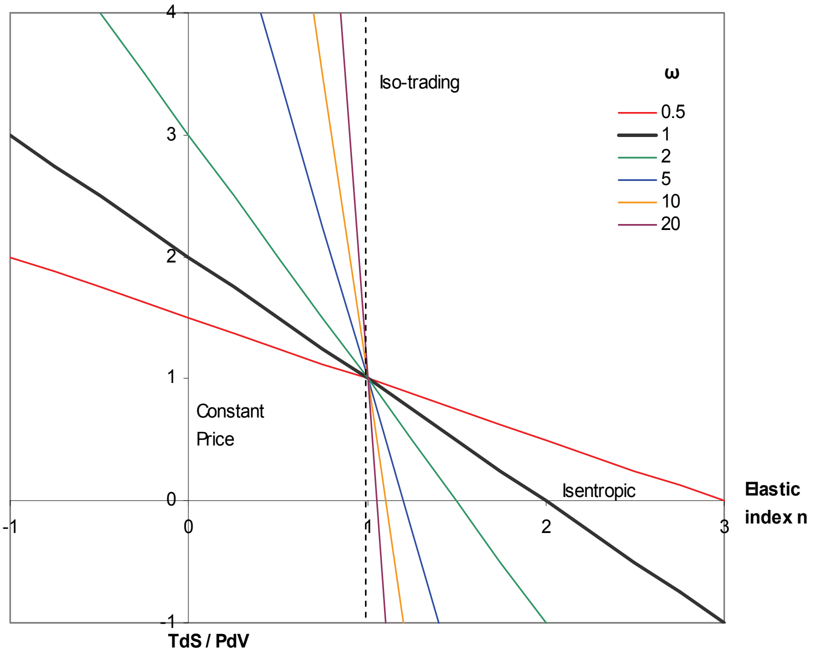

Figure 3.17 Marginal Entropic Ratio ***(ω – ωn + 1)*** *versus Elastic Index* ***n***

If marginal utility **∂Y** is taken to be equivalent to incremental change in entropic value **dQ (=TdS)**, *and* we are considering the same elastic relationship between price and volume and the same value capacity, then the Law of Diminishing Marginal Utility appears to be confirmed by thermodynamic analysis.

Finally, if we substitute **TdS** for **dQ** in equation (3.69), we have:

$$\left(\frac{TdS/dV}{P}\right) = (\omega - \omega n + 1)$$

Thence:

$$\left(\frac{dS/dV}{P}\right) = \left(\frac{1}{T}\right)(\omega - \omega n + 1) \qquad (3.71)$$

The same result is obtained if we consider a unit stock process, since entropy **S** and volume flow **V** are replace by **s=S/N** and **v=V/N**. Thus:

$$\left(\frac{ds/dv}{P}\right) = \left(\frac{1}{T}\right)(\omega - \omega n + 1) \qquad (3.72)$$

All of the forgoing analysis suggests that there is a strong relationship between the economic concept of utility **Y** and the thermodynamic concepts of entropic value **Q** and entropy **S,** with marginal utility **∂Y** being equivalent to the index of trading value **T** multiplied by incremental change in entropy **dS.**